CORRIGENDUM: THE SYMPLECTIC SUM FORMULA FOR GROMOV-WITTEN INVARIANTS

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Abstract. We correct an error and an oversight in [IP]. The sign of the curvature in (8.7) is wrong, requiring a new proof of Proposition 8.1. Several lemmas address only the basic case of maps with intersection multiplicity $s = \mathbf{1}$; the general case follows by applying the pointwise estimates in [IP] with a modified Sobolev norm.

1. Sobolev Norms

In [IP], portions of Sections 6-8 are valid only for maps with intersection multiplicity $s = \mathbf{1}$. To cover maps with multiplicity vector $s = (s_1, \ldots, s_\ell)$, we modify the Sobolev norms in [IP, (6.9)] by setting

$$\begin{aligned} \|\zeta\|^p_{m,p,s} \ &= \ \int_{C_\mu} \Big(|\nabla^m \zeta|^p + |\zeta|^p \Big)\ \rho^{-\delta p/2} \\ &\quad + \sum_k \int_{C_\mu \cap B_k(1)} \Big(|\rho^{1-s_k} \nabla^m(\zeta^N)|^p + |\rho^{1-s_k}\zeta^N|^p \Big)\ \rho^{-\delta p/2}. \end{aligned} \tag{1.1}$$

With this revision, the norms $\|\!|(\xi, h)\|\!|_m$ and $\|\!|\eta\|\!|_m$ are again defined by formulas [IP, (6.10)] and [IP, (6.11)]. For $s = \mathbf{1}$, the above norm is uniformly equivalent to the norm of [IP, (6.9)]. For general s, it now has a stronger weighting factor of ρ^{1-s_k} on the normal components near each node with multiplicity $s_k \geq 2$. Accordingly, one must verify that Lemmas 6.9 and 7.1, Proposition 7.3 and Lemma 9.2 continue to hold for this new norm. This is easily done using the pointwise estimates already appearing in the proofs, as follows.

Modifications to Section 6. Lemma 6.8 is unaffected by the change in norms. The statement of Lemma 6.9 remains valid for the new norms with a slight modification to the exponent in its conclusion:

$$\|\!|\overline{\partial}_{J_F} F - \nu_F\|\!|_0 \ \leq \ c\,|\lambda|^{\frac{1}{5|s|}}$$

with c uniform on each compact set $\mathcal{K}_{\delta_0}$. (Throughout [IP], the δ indexing the sets $\mathcal{K}_\delta$ of [IP, (3.11)] is unrelated to the exponent δ in (1.1).) The proof of Lemma 6.9 is modified as follows.

Proof of Lemma 6.9. Set $\Phi_F = \overline{\partial}_{J_F} - \nu_F$ and follow the proof in [IP] until (6.14). Outside $\rho \leq \rho_0$, the new norm is uniformly equivalent to old one, so [IP, (6.13)] continues to hold for $\Phi = \Phi_F$. Again we focus on the half A_+ of one A_k where $|w_k| \leq |z_k|$ where, after omitting subscripts, F is given by [IP, (6.14)]:

$$F = \big((1-\beta)h^v,\ az^s(1 + (1-\beta)h^x),\ bw^s(1+(1-\beta)h^x)^{-1}\big) \tag{1.2}$$

where β, h^v and h^x are functions of the coordinate z on $C_1 \subset C_0$ and $w = \mu/z$.

Introduce the map $\widetilde{f} : C_0 \to X$ given by (1.2) with the last entry replaced by zero. Noting that $\Phi_f = 0$ (because f is (J,ν)-holomorphic), we can write

$$\Phi_F \;=\; (\Phi_F - \Phi_{\tilde{f}}) + (\Phi_{\tilde{f}} - \Phi_f). \tag{1.3}$$

To complete the proof, we will bound the two expressions on the righthand side using the following facts, which hold for small λ:

(i) [IP, Lemma 6.8d] implies that $|h^v| + |dh^v| + |h^x| + |dh^x| \le c\rho \le \frac{1}{2}$ on A_+.

(ii) $|dz| = |z|$ and $|dw| = |w|$ in the cylindrical metric [IP, (4.5)], and $0 \le \beta \le 1$ and $|d\beta| \le 2$ by [IP, (5.11)].

(iii) On A_+, $\rho^2 = |z|^2 + |w|^2 \le 2|z|^2$, and hence $|z|^{-2} \le 2\rho^{-2}$ and $\sqrt{|\mu|} \le |z| \le \rho \le \rho_0$.

(iv) By [IP, Lemma 6.8c], a_k and b_k are bounded above and below by positive constants, and hence $|\mu_k^{s_k}| \sim |\lambda|$ by [IP, (6.3)].

Writing (1.2) as (F^v, F^x, F^y), Facts (i)-(iv) imply the pointwise bounds

$$|F^v| + |dF^v| \le c\rho, \qquad |F^x| + |dF^x| \le c\rho^s, \qquad |F^y| + |dF^y| \le c|w|^s \le C|\lambda|\rho^{-s} \tag{1.4}$$

for $s = s_k$, with constants c, C uniform on $\mathcal{K}_{\delta_0}$. It follows that $|dF| \le 3c\rho$ and, since J and ν are smooth,

$$|J_F - J_{\widetilde{f}}| + |\nu_F - \nu_{\widetilde{f}}| \;\le\; c|F - \widetilde{f}| \;=\; c|F^y| \;\le\; c|\lambda|\rho^{-s}. \tag{1.5}$$

(Here, and below, we are updating the constant c as we proceed.)

Now, using the definition of Φ_F, we have

$$2(\Phi_F - \Phi_{\tilde{f}}) \;=\; d(F - \tilde{f}) + (J_F - J_{\tilde{f}})dFj + J_{\tilde{f}}(dF - d\tilde{f})j - 2(\nu_F - \nu_{\widetilde{f}}), \tag{1.6}$$

with $F - \widetilde{f} = F^y$. Estimating each term, one sees that the above bounds imply that

$$|\Phi_F - \Phi_{\widetilde{f}}| \le c|\lambda|\rho^{-s} \quad \text{so} \quad \rho^{1-s}|(\Phi_F - \Phi_{\widetilde{f}})^N| \;\le\; c|\lambda|\rho^{1-2s} \tag{1.7}$$

on A_+. Applying the second integral in [IP, (5.10)], noting that $s \ge 1$ and that $\rho^2 \ge |\mu|$ on A_+ yields

$$|||\Phi_F - \Phi_{\widetilde{f}}|||_{0,A_+} \;\le\; c|\lambda||\mu|^{\frac{1}{2}(1-2s-\delta/2)} \;\le\; c|\lambda|^{\frac{1}{3s}} \tag{1.8}$$

where the last inequality uses (iv) above and the fact that $0 < \delta < \frac{1}{6}$. By symmetry, a similar estimate holds on the other half of A_k. Hence (1.8) holds on the entire set $A \subset C_0$ where $\rho \le \rho_0$ with a revised constant c and the exponent replaced by $\frac{1}{3|s|}$ (since $|s| \ge s_k$ for all k).

It remains to estimate the last term in (1.3). On A_+, the difference between f and $\widetilde{f}$, namely

$$f - \widetilde{f} = \beta(h^v, az^s(1 + h^x), 0),$$

is supported in the region $\rho \le 2|\mu|^{1/4}$. Again expanding as in (1.6) and using (i)-(iv), one obtains

$$|\Phi_{\widetilde{f}} - \Phi_f| \;\le\; |f - \widetilde{f}| + |df - d\widetilde{f}| + |J_f - J_{\widetilde{f}}| + |\nu_f - \nu_{\widetilde{f}}| \;\le\; c\rho$$

Similarly, using (i)-(iv), the normal component $(\widetilde{f} - f)^N = -\beta az^s(1 + h^x)$ satisfies:

$$|(\widetilde{f} - f)^N| + |d(\widetilde{f} - f)^N| \;\le\; c\rho^s.$$

Next observe that the images of f and $\widetilde{f}$ both lie in $Z_0 = X \cup Y$ and, as in [IP, (6.6)], J preserves the normal subbundle N_0 to V in Z along Z_0. Also noting that (1.4) implies that $|(d\widetilde{f})^N| \leq c\rho^s$, one sees that

$$\left|((J_f - J_{\widetilde{f}}) \circ d\widetilde{f})^N + (J_f(d\widetilde{f} - df))^N\right| \;\leq\; \left|(J_f - J_{\widetilde{f}}) \circ (d\widetilde{f})^N\right| + \left|J_f(d\widetilde{f} - df)^N\right| \;\leq\; c\rho^s.$$

Because ν^N vanishes along V, there is also a bound $|\nu^N_f| \leq C|f^N| \leq c\rho^s$; the same is true for $\widetilde{f}$, and therefore $|\nu^N_{\widetilde{f}} - \nu^N_f| \leq c\rho^s$. Expanding $\Phi_{\widetilde{f}} - \Phi_f$ as in (1.6) and using above estimates yields

$$|\Phi_{\widetilde{f}} - \Phi_f| + \rho^{1-s}|(\Phi_{\widetilde{f}} - \Phi_f)^N| \;\leq\; c\rho \tag{1.9}$$

with the lefthand side supported in the region $\rho \leq 2|\mu|^{1/4}$ in A_+ and, by symmetry, in A_k for each k. Applying the first integral in [IP, (5.10)] bounds the integrals in the norm (1.1) on the union A of the A_k. Again using the facts that $|\lambda| \sim |\mu_k|^{s_k}$, $0 < \delta < \frac{1}{6}$ and $s_k \leq |s|$, one obtains the bound

$$\|\Phi_{\widetilde{f}} - \Phi_f\|_{0,A} \;\leq\; c|\mu|^{\frac{1}{4}(1-\delta/2)} \;\leq\; c|\lambda|^{\frac{1}{5|s|}}. \tag{1.10}$$

The lemma now follows from [IP, (6.13)] and the bounds (1.8) and (1.10) on the norms of the two terms in (1.3). □

Modifications to Section 7. Delete the paragraph that starts after [IP, (7.4)] and ends with [IP, (7.6)]; we no longer need D^*_F.

- The conclusion of Lemma 7.2 should read

$$|(\nabla\zeta^V)^N| \leq c\rho^s|\zeta^V|, \qquad |L^N_F\zeta^V| \leq c\rho^s|\zeta^V| \tag{1.11}$$

as the proof shows (keep all powers of ρ in the last line of the two paragraphs of the proof).

- The statement of Proposition 7.3 remains the same after deleting the statement about D^*_F. The proof is unchanged until two lines before [IP, (7.10)], at which point we have established the estimate

$$|\mathbf{D}_{F,C}(\zeta, \overline{\xi}, h)| \;\leq\; c|\nabla\zeta| + c\rho(|\zeta| + |\overline{\xi}| + |h|) \tag{1.12}$$

(this is also [IP, (9.7)]). As a special case, we have

$$|\mathbf{D}_{F,C}(\zeta^N, 0, 0)^N| \;\leq\; c\big(|\nabla\zeta^N| + \rho|\zeta^N|\big).$$

On the other hand, the normal component is

$$\big(\mathbf{D}_{F,C}(\zeta^V, \overline{\xi}, h)\big)^N \;=\; \Big(L_F(\zeta^V + \sum \beta_k\overline{\xi}_k)\Big)^N + (J_F dFh)^N.$$

Using (1.11), the first term on the right is bounded by $c\rho^s(|\zeta^V| + |\overline{\xi}|)$. The second is dominated by $|(J_F(dF^V + dF^N))^N|\,|h|$ with $|dF^N| \leq c\rho^s$ by (1.4). Furthermore, because V is J-holomorphic, $(Jv)^N = 0$ along V for all vectors v in the V direction. Hence $|(J_F dF^V)^N| \leq c|F^N||dF^V| \leq c\rho^s$, again using (1.4). Altogether, this gives the following pointwise bound:

$$\left|\mathbf{D}_{F,C}(\zeta, \overline{\xi}, h)^N\right| \;\leq\; c\left(|\nabla\zeta^N| \,+\, \rho|\zeta^N|\right) + c\rho^s\left(|\nabla\zeta^V| + |\zeta^V| + |\overline{\xi}| + |h|\right). \tag{1.13}$$

Multiplying both sides of this inequality by $\rho^{1-s-\delta/2}$, raising to the power p and integrating shows that [IP, (7.10)] holds in the new norms. The proof is completed as before.

Modifications to Section 8. See Section 2 below.

Modifications to Section 9. Section 9 uses Proposition 8.1, but not its proof: the existence of a right inverse is needed, but nothing about its construction. Switching to the new norms (1.1) does not affect Proposition 9.1, and requires only small modifications to the proofs of Lemma 9.2 and Proposition 9.3.

• **Lemma 9.2**: The statement of Lemma 9.2 remains the same. For the proof, again let B_k be the region around the kth node where $\rho \le |\mu|^{1/4}$ and let A_k be the larger region where $\rho \le \rho_0$. The new norms (1.1) differ from the old norms only in the weighting of the normal components near the nodes. Thus to show that Lemma 9.2 holds in the new norms we need only bound the L^p integral of the weighted normal components

$$|\rho^{1-s_k-\delta/2}\mathbf{D}_F(\xi,h)^N|,$$

first on $A_k \setminus B_k$, then on B_k. Fix k and write $\mu = \mu_k$ and $s = s_k$.

Follow the existing proof until two lines below [IP, (9.5)], at which point we have identified sections of F^*TZ_λ over $C_\mu \setminus B$ with sections of f^*TZ_0 over $C_0 \setminus B$, and established the estimate:

$$(1.14)\qquad |D_F(\xi,h) - D_f(\xi,h)| \;\le\; |(L_F - L_f)(\xi)| + (|J_F - J_f| + |dF - df|)|h|$$

with $D_f(\xi,h) = 0$. Formula [IP, (1.11)] shows that L_f is a first order differential operator of the form

$$L_f(\xi) \;=\; A_f(\nabla\xi) + B_f(\xi)$$

whose coefficients A_f and B_f are continuous functions of f and df. Hence

$$(1.15)\qquad |L_F\xi - L_f\xi| \;\le\; c\,(|F-f| + |dF - df|)\cdot(|\nabla\xi| + |\xi|)$$

for some constant c. But in the region $A_k \setminus B_k$ we have $F - f = F - \widetilde{f} = F^y$. Then (1.4) shows that the C^1 distance between F_t and f_t is dominated by $|\lambda|\rho^{-s}$, so (1.14) implies the bound

$$|D_F(\xi,h)| \;\le\; c|\lambda|\rho^{-s}\,(|\nabla\xi| + |\xi| + |h|)$$

on $A_k \setminus B_k$. After expanding ξ as in [IP, (6.8)] and noting that $|\nabla(\beta_k\xi_k)| \le c|\overline{\xi}|$ by the estimate preceding [IP, (7.9)], the above bound simplifies to

$$(1.16)\qquad |\mathbf{D}_F(\zeta,\overline{\xi},h)| \;\le\; c|\lambda|\rho^{-s}\left(|\nabla\zeta| + |\zeta| + |\overline{\xi}| + |h|\right),$$

a mild strengthening of the displayed equation above [IP, (9.6)]. But in the region $A_k \setminus B_k$, we have $|\mu|^{1/4} \le \rho \le \rho_0$ and $|\lambda| \sim |\mu|^s$ so (1.16) implies

$$(1.17)\qquad |\mathbf{D}_F(\zeta,\overline{\xi},h)| + \rho^{1-s}|\mathbf{D}_F(\zeta,\overline{\xi},h)^N| \;\le c|\lambda|^{1/4}\rho^{1+s}\left(|\nabla\zeta| + |\zeta| + |\overline{\xi}| + |h|\right).$$

Taking the norms defined by [IP, (6.10)] and (1.1), shows that [IP, (9.6)] continues to hold in the new norms.

Now focus on one B_k. Proceed as in [IP], using the new estimate (1.13) to strengthen [IP, (9.7)]. For $\xi = \xi^V$, (1.13) gives

$$(1.18)\qquad |\mathbf{D}_F(\zeta^V,\overline{\xi},h)^N| \;\le\; c\rho^s\left(|\nabla\zeta^V| + |\zeta^V| + |\overline{\xi}| + |h|\right)$$

on each B_k. For $\xi = \xi^N$, we again have $\overline{\xi} = 0$ and $\xi = \zeta$, so (1.13) and the last displayed equation on page 988 give

$$(1.19)\qquad |\mathbf{D}_F(\xi^N,0)^N| \;\le\; c\left(|\nabla\zeta^N| + \rho|\zeta^N|\right) \;\le\; c\rho^s(|\dot{a}| + |\dot{b}|).$$

Combining (1.18) and (1.19) with the argument on top of page 989 shows that the conclusion of the first displayed equation on top of page 989 continues to hold in the new norms. The proof is completed as before.

• **Proposition 9.3**: Replace the last 4 lines on page 989 of [IP] by the following: write $F_n - f_n = (\zeta_n, \bar{\xi}_n)$ in the notation of (6.7) and (6.8). Then $\bar{\xi}_n \to 0$ because $f_n \to f_0$ in C^0. By Lemma 5.4, the norm $|||f_n|||_1$ on $A_k(\rho_0)$ is bounded by $c\rho_0^{1/6}$. Inserting the bounds (1.4) into (1.1) and integrating using [IP, (5.10)] gives the similar inequality $|||F_n|||_1 \le c\rho_0^{1/6}$ on $A_k(\rho_0)$. Therefore $|||\zeta_n|||_1 \le |||F_n|||_1 + |||f_n|||_1 + |\bar{\xi}_n| \le 3c\rho_0^{1/6}$ on $A_k(\rho_0)$ for all large n. Combining ...
Continue at the top of page 990, and change $2|s| \mapsto 5|s|$ *on page 990, line 14.*

2. **Revised Section 8**

An incorrect formula [IP, (7.5)] for the adjoint and a sign error in the curvature formula [IP, (8.7)] invalidate the proof of Proposition 8.1. The following replacement for Section 8 retains everything up to and including the statement of Proposition 8.1, and then gives a new proof of Proposition 8.1. Instead of establishing eigenvalue estimates, this new proof transfers the partial right inverse P from the nodal curve C_0 to its smoothing C_μ. The proof is then easier, the adjoint D_F^* never appears, and again the required estimates follow from pointwise bounds already in [IP].

Retain the beginning of Section 8 of [IP] up to Proposition 8.1.

To simplify notation, note that for $F \in \mathcal{A}pprox_s^{\delta_0}(Z_\lambda)$, [IP, Proposition 7.3] shows that the linearizations $\mathbf{D}_F$ of [IP, (7.4)] are uniformly bounded operators

$$\mathbf{D}_F : \mathcal{E}_F \to \mathcal{F}_F$$

between the spaces

$$\mathcal{E}_F = L_{1;s,0}(F^*TZ_\lambda) \oplus T_qV^\ell \oplus T_{C_\mu}\overline{\mathcal{M}}_{g,n} \quad \text{and} \quad \mathcal{F}_F = L_s(\Lambda^{01}(F^*TZ_\lambda)),$$

while the linearization $\mathbf{D}_f = \mathbf{D}_{f,C_0}$ of [IP, (7.3)] at each $f \in \mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$ is a map between the corresponding spaces $\mathcal{E}_f$ and $\mathcal{F}_f$.

The aim of this section is to prove the following analytic result.

Proposition 2.1. *For each generic $(J,\nu) \in \mathcal{J}(Z)$, there are positive constants λ_0 and E such that, for all non-zero $\lambda \le \lambda_0$, the linearization $\mathbf{D}_F$ at an approximate map $(F, C_\mu) = F_{f,C_0,\mu} \in \mathcal{A}pprox_s^{\delta_0}(Z_\lambda)^*$ has a right inverse*

$$P_F : \mathcal{E}_F \to \mathcal{F}_F \tag{2.20}$$

that satisfies $\mathbf{D}_F P_F = id$ and

$$E^{-1}\, |||\eta|||_0 \;\le\; |||P_F\eta|||_1 \;\le\; E\, |||\eta|||_0. \tag{2.21}$$

Proposition 8.1 is proved by constructing an approximation to P_F in the following sense.

Definition 2.2. *An* approximate right inverse to $\mathbf{D}_F$ *is a linear map*

$$A_F : \mathcal{F}_F \to \mathcal{E}_F$$

such that, for all $\eta \in \mathcal{F}_F$,

$$|||D_F A_F\eta - \eta|||_0 \;\le\; \tfrac{1}{2}|||\eta|||_0 \qquad \textit{and} \qquad |||A_F\eta|||_1 \;\le\; C|||\eta|||_0. \tag{2.22}$$

Such an approximate right inverse defines an actual right inverse by the formula

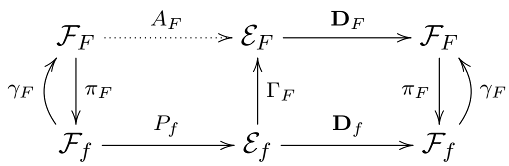
$$P_F = A_F \sum_{k\geq 0} (I - D_F A_F)^k.$$

The bounds (2.22) ensure that this series converges and defines a bounded operator P_F, which satisfies $D_F P_F = I$. Because both P_F and D_F are bounded (cf. [IP, (Lemma 7.3)]), we have $\|P_F\eta\|_1 \ \leq\ c\,\|\eta\|_0$ and $\|\eta\|_0 = \|D_F P_F \eta\|_0 \leq c\,\|P_F\eta\|_1$, which gives (2.21).

Thus Proposition 8.1 follows from the existence of an approximate right inverse A_F as in Definition 2.2, where the constant C in (2.22) is uniform in λ for small λ. The remainder of this section is devoted to constructing such an A_F.

We start by observing that, under the hypotheses of Proposition 2.1, we may assume that f is regular (cf. [IP, (Lemma 3.4)]). Thus $\mathbf{D}_f : \mathcal{E}_f \to \mathcal{F}_f$ is a bounded surjective map, so has a bounded right inverse $P_f : \mathcal{F}_f \to \mathcal{E}_f$. We will use a splicing construction to transfer P_f from an operator on C_0 to one on the domain C_μ of F, and show that the resulting operator A_F satisfies (2.22). The construction is summarized by the following (noncommutative) diagram:

(2.23)



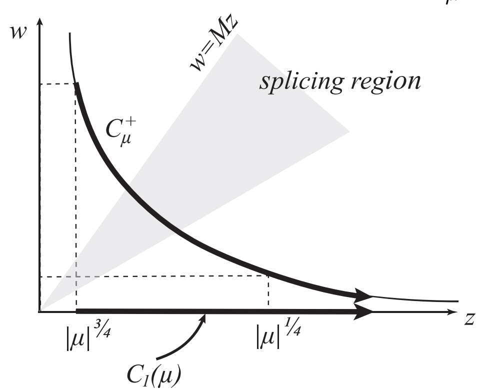
Each of the maps γ_F, π_F and Γ_F will be defined by regarding the two halves of C_μ as a graphs over C_0, and similarly regarding Z_λ as graphs over Z_0. The desired approximate right inverse is then defined by

$$A_F = \Gamma_F \circ P_f \circ \pi_F.$$

Our notation for splicing is as in Lemma 9.2 of [IP]. For each $\mu \neq 0$, let

$$C_1(\mu) \ =\ C_1 \cap \left\{ |z| \geq |\mu|^{\frac{3}{4}} \right\} \qquad C_2(\mu) \ =\ C_2 \cap \left\{ |w| \geq |\mu|^{\frac{3}{4}} \right\}.$$

and let C_μ^+ and C_μ^- be the corresponding parts of C_μ (see the figure). We identify $C_1(\mu)$ with C_μ^+ by the projection $(z, w) \mapsto z$. With this identification, z is a coordinate on both $C_1(\mu)$ and C_μ^+ and, similarly, w is a coordinate on both $C_2(\mu)$ and C_μ^-.



There is a corresponding picture in the target: the projections $(v, x, y) \mapsto (v, x)$ and $(v, x, y) \mapsto (v, y)$ give identifications $Z_\lambda = X$ in the region Z_λ^+ where $|x| \geq |\lambda|^{3/4}$, and $Z_\lambda = Y$ in the region Z_λ^- where $|y| \geq |\lambda|^{3/4}$. This trivializes f^*TZ_0 and F^*TZ_λ inside the coordinate chart (v, x, y).

These identifications, together with [IP, Definition 6.2], induce isomorphisms

$$\Omega^{0,q}(C_1(\mu), f^*TX) \to \Omega^{0,q}(C^+_\mu, F^*TZ_\lambda) \qquad \text{by } \xi_1 \mapsto \widehat{\xi}_1 \tag{2.24}$$

for $q = 0, 1$ defined by $\widehat{\xi}_1(z) = \xi(z)$ in B_k under the above identifications of $C_1(\mu)$ with C^+_μ and of X with Z^+_λ, extended by setting $\widehat{\xi}_1 = \xi$ outside the union of the balls B_k of radius $2|\mu_k|^{1/4}$ (where C_0 is identified with C_μ and $F = f$). Permuting $z \leftrightarrow w$ and $x \leftrightarrow y$ gives similar isomorphisms

$$\Omega^{0,q}(C_2(\mu), f^*TX) \to \Omega^{0,q}(C^-_\mu, F^*TZ_\lambda) \qquad \text{by } \xi_1 \mapsto \widehat{\xi}_2.$$

Lemma 2.3. *For each region Ω^+_M defined by $M^{-1}|w| \le |z| \le 1$, there are constants $c_M, \lambda_M > 0$ such that the map (2.24) satisfies the pointwise estimates*

$$c_M^{-1}|\xi| \;\le\; |\widehat{\xi}| \;\le\; c_M|\xi|, \qquad |\nabla\widehat{\xi}| \;\le\; c_M\big(|\nabla\xi| + |\xi|\big) \tag{2.25}$$

whenever $|\lambda| \le \lambda_M$ small. Furthermore, if $\xi = \xi^V$ then $(\widehat{\xi}\,)^N = 0$.

Proof. For each non-zero small μ, equations [IP, (4.4), (4.5)] show that the cylindrical metric on $C_\mu \cap \Omega^+_M$ is independent of μ (the ratio g_μ/g_0 of the metrics is $r^2\rho^{-2}\left(1 + |\frac{\mu}{z}|^2\right) = 1$). On the target, the corresponding formula shows that the (smooth) metrics g_X on X and g_λ on Z_λ have the form

$$g_X \;=\; (g_V + dxd\bar{x}) + O(R) \qquad g_\lambda \;=\; \left[g_V + \left(1 + \tfrac{|\lambda|^2}{|x|^4}\right) dxd\bar{x}\right] + O(R), \tag{2.26}$$

where g_V is the metric on V, and $R^2 = |x|^2 + |y|^2$. Using the formula for the Christoffel symbols, one sees that the difference of the corresponding Levi-Civita connections is a 1-form $\alpha\, dx$ on X with $|\alpha| \le c(1 + |\lambda|^2|x|^{-5})$.

As in (1.2)–(1.4), the coordinates of each approximate map F satisfy $|x| \sim |z|^s$ and $|y| \sim |w|^s$ and $xy = \lambda$. Thus the image of Ω^+_M lies in the region Z_λ where $|y| \le c_1(M)|x|$ for a constant $c_1(M)$ independent of λ. In this region, the metrics (2.26) are uniformly equivalent with a similar constant $c_2(M)$, giving the first part of (2.25). Furthermore, the covariant derivatives are related by

$$\nabla\widehat{\xi} = \nabla\xi + \alpha_F\xi \qquad \text{with } |\alpha_F\xi| \;\le\; c_3(M)\,(1 + |\lambda|^2|x|^{-5}) \cdot \big(|df^N| + |dF^N|\big)\; |\xi|.$$

Noting that $xy = \lambda$ and $|x| \sim |z|^s \sim \rho^s$, the term $|\lambda|^2|x|^{-5}$ is dominated by $\left|\frac{y}{x}\right|^2 \rho^{-s} \le c_1^2(M)\rho^{-s}$. We also have $|dF^N| \le c\rho^s$ by (1.4) and the bound $|df^N| \le c\rho^s$ obtained similarly by taking $\beta = 0$ in (1.2). The last part of (2.25) follows. □

To define cutoff functions, consider the central annular region Ω_M of C_μ defined by

$$M^{-1} \le \left|\frac{w}{z}\right| \;\le\; M. \tag{2.27}$$

In cylindrical coordinates (defined by $z = \sqrt{|\mu|}e^{t+i\theta}$), this is a region of length $\log(1 + M^2)$. Thus we can choose a smooth cutoff function $\varphi_M(z, w)$ that vanishes for $|w| > M|z|$, is equal to 1 for $M|w| \le |z|$, and satisfies $0 \le \varphi_M \le 1$ and

$$|d\varphi_M| \le \frac{1}{|\log M|}. \tag{2.28}$$

To maintain symmetry, we can also assume (after appropriately symmetrizing) that

$$\varphi_M(z,w)+\varphi_M(w,z)=1.$$

With this setup, the maps γ_F, π_F and Γ_F in Diagram (2.23) are defined as follows.

• **The map** $\pi_F:\mathcal{F}_F\to\mathcal{F}_f$**.** The map (2.24) with $q=1$ has an inverse

$$\tau^+:\Omega^{0,1}(C_\mu^+,F^*TZ_\lambda)\to\Omega^{0,1}(C_1(\mu),f^*TX).$$

Then each F^*TZ_λ-valued (0,1)-form η on C_μ restricts to a form η^+ on C_μ^+, and we define $\pi_F(\eta)$ on C_1 by

$$(\pi_F(\eta))(z)=\begin{cases}\tau^+\eta^+(z) & \text{for } |z|>|\mu|^{1/2}\\ 0 & \text{for } |z|\le|\mu|^{1/2}.\end{cases}$$

The restriction of $\pi_F(\eta)$ to C_2 is defined symmetrically.

• **The map** $\gamma_F:\mathcal{F}_f\to\mathcal{F}_F$**.** This map takes a f^*TZ_0-valued (0,1)-form η on C_0 to a F^*TZ_λ-valued (0,1)-form η on C_μ. It is given by

$$\gamma_F(\eta)\;=\;\varphi_M\,\widehat{\eta}_1\;+\;(1-\varphi_M)\,\widehat{\eta}_2 \tag{2.29}$$

where $\widehat{\eta}_1$ and $\widehat{\eta}_2$ are defined in terms of the restrictions $\eta|_{C_1}=\xi_1\,d\bar z$ and $\eta|_{C_2}=\xi_2\,d\bar w$ by $\widehat{\eta}_1=\widehat{\xi}_1\,d\bar z$ and $\widehat{\eta}_2=\widehat{\xi}_2\,d\bar w$ inside each coordinate chart (z,w), and $\gamma_F=id$ outside.

• **The map** $\Gamma_F:\mathcal{E}_f\to\mathcal{E}_F$**.** This map takes a section of f^*TZ_0 on C_0 to a section of F^*TZ_λ on C_μ, and a variation h in the complex structure of C_0 to a variation in the complex structure of C_μ. It is given by

$$\Gamma_F(\xi,h_0)\;=\;\big(\varphi_M\,\widehat{\xi}_1+(1-\varphi_M)\,\widehat{\xi}_2,\;h_\mu\big) \tag{2.30}$$

where $\widehat{\xi}_1$ and $\widehat{\xi}_2$ are obtained from the restrictions $\xi|_{C_1}=\xi_1$ and $\xi|_{C_2}=\xi_2$ inside these neighborhoods, and $h_\mu=(h_0,0)$ in the notation of [IP, (4.9)]. Again, Γ_F extends outside as $\Gamma_F=id$. Thus, in the notation of [IP, (7.3), (7.4)], Γ_F is a map

$$\Gamma_F:L_{1;s}(f^*TZ_0)\oplus T_{C_1}\widetilde{\mathcal{M}}\oplus T_{C_2}\widetilde{\mathcal{M}}\to L_{1;s}(F^*TZ_\lambda)\oplus T_{C_\mu}\overline{\mathcal{M}}_{g,n}.$$

Corollary 2.4. *The maps π_F, γ_F and Γ_F satisfy*

$$\pi_F\gamma_F=id,$$

and for small λ

$$|||\pi_F\eta|||_0\le 2|||\eta|||_0\qquad |||\gamma_F\eta|||_0\le c_M|||\eta|||_0\qquad |||\Gamma_F(\xi,h)|||_1\le c_M|||(\xi,h)|||_1, \tag{2.31}$$

where c_M depends only on the constant M in (2.27).

Proof. The equation $\pi_F\gamma_F=id$ follows directly from the definitions of π_F and γ_F. As in first paragraph of the proof of Lemma 2.3, the projection $C_\mu\to C_1$ is an isometry in the region where $\rho\le1$, and g_λ is greater than g_X on its image. It follows that the operator norm of π_F is at most 2 for small λ.

Similarly, (2.29), the fact that $0\le\varphi_M\le1$, and Lemma 2.3 show that

$$|||\gamma_F(\eta)|||_0\;\le\;|||\widehat{\eta}_1|||_0\;+|||\widehat{\eta}_2|||_0\;\le\;c_M|||\eta|||.$$

Using (2.30) in exactly the same way, we also have

$$\|\Gamma_F(\xi,h)\|_0 \;\le\; \|\widehat{\xi}_1\|_0 \;+\|\widehat{\xi}_2\|_0+\|h\| \;\le\; c_M\|(\xi,h)\|_0.$$

Differentiating (2.30) and again applying Lemma 2.3, yields the last inequality in (2.31). □

The next lemma shows that the difference $\mathbf{D}_F\Gamma_F-\gamma_F\mathbf{D}_f$ can be made small. The statement again involves the constant M in the bounds (2.27) and (2.28) associated with the cutoff functions φ_M.

Lemma 2.5. *Fix the compact subset $\mathcal{K}_{\delta_0}$ of $\mathcal{M}_s^V(X)\times_{\rm ev}\mathcal{M}_s^V(Y)$ of δ_0-flat maps. For any $\varepsilon>0$, there exits a slope $M=M_\varepsilon>1$ and a $\lambda_M>0$ such that each approximate map F constructed from $f\in\mathcal{K}_{\delta_0}$ with $|\lambda|\le\lambda_M$ satisfies*

$$\|(\mathbf{D}_F\Gamma_{F,M}-\gamma_F\mathbf{D}_f)(\xi,h)\|_0 \;\le\; \varepsilon\|(\xi,h)\|_1$$

for all $(\xi,h)\in\mathcal{E}_f$.

Proof. We use the set-up of Lemma 9.2 above, except that we do not make the assumption that $\mathbf{D}_f(\xi_0,h_0)=0$ in (1.14). Outside the region $B=\bigcup B_k$, $\Gamma_{F,M}$ and γ_F are both the identity for $|\mu|\le M^{-2}$. Thus the discussion from (1.14) to (1.17) implies the bound

$$\|(\mathbf{D}_F\Gamma_F-\gamma_F\mathbf{D}_f)(\xi,h)\|_{0,C_0\setminus B} \;\le\; c|\lambda|^{\frac14}\;\|(\xi,h)\|_1.$$

Next restrict attention to the region B_k around one node of C_0, where $\rho\le 2|\mu|^{1/4}$, and consider a deformation (ξ,h) on C_1 (an identical analysis applies on C_2). Then $\xi\in\Gamma(C_1,f^*TX)$ lifts by (2.24) to $\widehat{\xi}\in\Gamma(C_\mu^+,F^*TZ_\lambda)$: this is the identification implicitly used in [IP, (9.5)] and in (1.14). With this notation, (1.14) can be written as

$$|\mathbf{D}_F(\widehat{\xi},h)-\widehat{\mathbf{D}_f(\xi,h)}| \;\le\; |L_F\widehat{\xi}-\widehat{L_f\xi}|+c|J_F-J_f||\xi|+|dF-df||h|.$$

For the following estimates, we restrict attention to the annular subregion $A_M^+\subset B_k$ where

$$|w/z|\le M \qquad\text{and thus}\qquad |z|\le\rho\le|z|\sqrt{1+M^2}. \tag{2.32}$$

In this subregion, the C^1 norm of $F-f$ is $O(\rho)$, as shown in the proof of Lemma 6.9. Using (1.14) and (1.15) we obtain

$$|\mathbf{D}_F(\widehat{\xi},h)-\widehat{\mathbf{D}_f(\xi,h)}| \;\le\; c\rho\big(|\nabla\widehat{\xi}|+|\widehat{\xi}|+|\nabla\xi|+|\xi|+|h|\big).$$

Lemma 2.3 then shows that we may remove the hats on the righthand side, giving

$$|\mathbf{D}_F(\widehat{\xi},h)-\widehat{\mathbf{D}_f(\xi,h)}| \;\le\; c\rho\big(|\nabla\xi|+|\xi|+|h|\big). \tag{2.33}$$

Here the left-hand side is regarded as a function of z and $w=\mu/z$ on $C_\mu\cap A_M^+$, while ξ and h are functions of z on the corresponding region of C_1, and $\rho^2=|z|^2+|w^2|$.

As in previous lemmas, we need separate bounds for the normal components. First, (2.33) for $\xi=\xi^N$ and $h=0$ gives

$$\Big|\mathbf{D}_F(\widehat{\xi^N},0)^N-\mathbf{D}_f\widehat{(\xi^N,0)}\Big| \;\le\; c\rho\big(|\nabla\xi^N|+|\xi^N|\big). \tag{2.34}$$

On the other hand, if $\xi=\xi^V$ is tangent to V, Lemma 2.3 shows that its lift $\widehat{\xi}$ is also tangent to V. Writing $D_F(\xi,h)=L_F\xi+J_FdFh$, we can apply (1.11) and the argument made before (1.13) to obtain

$$|\mathbf{D}_F(\widehat{\xi^V},h)^N| \;\le\; c\rho^s\left(|\nabla\widehat{\xi^V}|+|\widehat{\xi^V}|+|h|\right).$$

By the same argument, a similar inequality holds with D_F replaced by D_f and $\widehat{\xi^V}$ by ξ^V; together these give

$$|\mathbf{D}_F(\widehat{\xi^V},h)^N - (\widehat{\mathbf{D}_f(\xi^V,h)})^N| \;\le\; c\rho^s\left(|\nabla\xi^V| + |\xi^V| + |h|\right)$$

after again using (2.25) to remove hats on the right. Combining this with (2.34) gives

$$(2.35)\quad |\mathbf{D}_F(\widehat{\xi},h)^N - (\widehat{\mathbf{D}_f(\xi,h)})^N| \;\le\; c\rho(|\nabla\xi^N| + |\xi^N|) + c\rho^s\left(|\nabla\xi^V| + |\xi^V| + |h|\right).$$

Now set $\Psi(\xi,h) = D_F(\widehat{\xi},h) - \widehat{\mathbf{D}_f(\xi,h)}$. With this notation, we can combine (2.33) and (2.35), and then decompose ξ into $(\zeta,\bar{\xi})$ as in [IP, (6.9)], noting that $|\nabla\xi| + |\xi| \le |\nabla\zeta| + |\zeta| + |\bar{\xi}|$ as before (1.16). The result is

$$\begin{aligned} |\Psi(\xi,h)| + \rho^{1-s}|(\Psi(\xi,h))^N| &\le c\rho\Big(|\nabla\xi| + |\xi| + |h| + \rho^{1-s}(|\nabla\xi^N| + |\xi^N|)\Big) \\ &\le c\rho\Big(|\nabla\zeta| + |\zeta| + |\bar{\xi}| + |h| + \rho^{1-s}(|\nabla\xi^N| + |\xi^N|)\Big). \end{aligned}$$

Multiplying by $\rho^{-\delta/2}$ and computing the integral (1.1) over $A_M^+ \cap C_\mu$, where $\rho \sim |z|$ by (2.32) and $|\lambda| \sim |\mu|^s$, one sees that

$$\begin{aligned} \|\!|\mathbf{D}_F(\widehat{\xi_1},h_1) - \widehat{\mathbf{D}_f(\xi_1,h_1)}|\!\|_{0,A_M^+\cap C_\mu} &\le c_M|\mu|^{1/4(1-\delta/2)}\|\xi_1,h_1\|_1 \\ (2.36)\qquad &\le c_M|\lambda|^{\frac{1}{5|s|}}\|\xi_1,h_1\|_1 \end{aligned}$$

for all pairs (ξ_1,h_1) on C_1. A similar estimate holds for pairs (ξ_2,h_2) on C_2.

To complete the proof, recall that both Γ_F and γ_F are obtained by splicing in the region Ω_M of (2.27). Using (2.30), the formula $\mathbf{D}_F(\xi,h) = L_F\xi + JdFh$ and the Leibniz rule, we obtain

$$\Big|\mathbf{D}_F\Gamma_F(\xi,h) - \varphi_M\mathbf{D}_F(\widehat{\xi_1},h_\mu) - (1-\varphi_M)\mathbf{D}_F(\widehat{\xi_2},h_\mu)\Big| \;\le\; |d\varphi_M|\cdot|\xi| \;\le\; |\log M|^{-1}|\xi|.$$

Combining this with (2.29) and (2.36) gives the following uniform estimate:

$$\|\!|(\mathbf{D}_F\Gamma_{F,M} - \gamma_F\mathbf{D}_f)(\xi,h)|\!\|_0 \;\le\; \left(c_M\lambda^{\frac{1}{5|s|}} + c|\log M|^{-1}\right)\|\!|(\xi,h)|\!\|_1.$$

The lemma follows by first choosing M so that $c|\log M|^{-1} \le \varepsilon/2$, then choosing λ_M so that $c_M\lambda_M^{\frac{1}{5|s|}} \le \varepsilon/2$. $\square$

We are now able to define the approximate inverse of $\mathbf{D}_F$. Recall that the linearization $\mathbf{D}_f$ of a regular map is onto. Fix the compact set $\mathcal{K} = \mathcal{K}_{\delta_0}$ of δ_0-flat regular maps. Then we can choose a family P_f of partial right inverses of D_f which are uniformly bounded

$$\|P_f\eta\|_0 \;\le\; K\|\eta\|_1.$$

on $\mathcal{K}$. Recall that the operator norm of π_F is at most 2.

Lemma 2.6. *In the above context, there exist positive constants C, M and λ_0 such that for any approximate map F (obtained from $f \in \mathcal{K}$ for $|\lambda| \le \lambda_0$), the operators*

$$A_F = \Gamma_{F,M} \circ P_f \circ \pi_F : \mathcal{E}_F \to \mathcal{F}_F$$

obtained from splicing in region (2.27) *satisfy*

$$(2.37)\qquad \|\!|D_FA_F\eta - \eta|\!\|_0 \;\le\; \tfrac{1}{2}\|\!|\eta|\!\|_0 \qquad \text{and} \qquad \|\!|A_F\eta|\!\|_1 \;\le\; C\|\!|\eta|\!\|_0$$

for all $\eta \in \mathcal{F}_F$.

Proof. Write $D_F A_F - I$ as

$$\begin{aligned} D_F \circ \Gamma_F \circ P_f \circ \pi_F - id &= (D_F\Gamma_F - \gamma_F D_f) \circ P_f \circ \pi_F + \gamma_F D_f \circ P_f \circ \pi_F - id \\ &= (D_F\Gamma_F - \gamma_F D_f) \circ P_f \circ \pi_F. \end{aligned}$$

We know that $|||\pi_F||| \leq 2$ and $|||P_f||| \leq K$ are uniformly bounded on the compact $\mathcal{K}$. Hence there is a bound on the operator norm:

$$\|D_F A_F - I\| \ \leq\ \|D_F\Gamma_F - \gamma_F D_f\| \cdot \|P_f\| \cdot \|\pi_F\| \ \leq\ 2K\|D_F\Gamma_F - \gamma_F D_f\|.$$

Now take $\varepsilon = \frac{1}{4K} > 0$ in Lemma 2.5 to obtain the first inequality of (2.22). This choice of ε fixes the slope $M = M_\varepsilon$ in Lemma 2.5. With this choice, $\Gamma_{F,M}$ are bounded, with a bound that depends on M, and hence

$$\|A_F\| \ =\ \|\Gamma_{F,M} \circ P_f \circ \pi_F\| \ \leq\ \|\Gamma_{F,M}\| \cdot \|P_f\| \cdot \|\pi_F\| \ \leq\ 2K\|\Gamma_{F,M}\|.$$

□

3. **Typographical errors**

The following typographical errors in [IP] have no consequences, but may cause confusion.

- Line after (5.1): insert "after passing to a subsequence".
- In (5.18), delete $+1/3$.
- In (1.11), the second $+$ should be a $-$.
- Page 946, line 4: $J(\nabla_{\nu(w)}J)\xi \to (\nabla_{J\nu(w)}J)\xi$. One can also note that the tensor $\widehat{\nabla} J$ is zero by (C.7.5) of [MS] for compatible structures (ω, J, g).
- Page 1003, line 8: $(10.6) \mapsto (10.11)$.

References

[IP] E.N. Ionel and T.H. Parker, *The Symplectic Sum Formula for Gromov-Witten Invariants*, Annals of Math. **159** (2004), 935-1025.

[MS] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology* American Mathematical Society Colloquium Publications, **52**, AMS, Providence, RI, 2004.

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